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ON SOFIC SYSTEMS II

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ABSTRACT
The results of *On sofic systems I* on topological Markov chains extending sofic systems are completed. To homomorphisms of sofic systems are canonically associated homomorphisms of Markov extensions. Also considered is a class of finitary codes for sofic systems.

1. Introduction

This paper is the sequel and conclusion of [4]. It contains some remarks that complement and complete the results of [4]. We continue to use notation and terminology as in [4].

In [4] we considered for a sofic system (Y, S_Σ) an extension

$$\rho_+(Y, S_\Sigma) : (X_+(Y, S_\Sigma),\ S_+(Y, S_\Sigma)) \to (Y, S_\Sigma),$$

and for a topologically transitive sofic system (Y, S_Σ) with periodic points dense also an extension

$$\rho_+^0(Y, S_\Sigma) : (X_+^0(Y, S_\Sigma), S_+^0(Y, S_\Sigma)) \to (Y, S_\Sigma).$$

The canonicity of $\rho_+^0(Y, S_\Sigma)$ was shown in [4]. In the meantime a different proof was given by M. Boyle, B. Kitchens and B. Marcus [1]. In Section 2 we show that also $\rho_+(Y, S_\Sigma)$ is canonical.

In Section 3 we consider homomorphisms. In particular, we point out that to homomorphisms between sofic systems there are canonically associated topological Markov chains, and homomorphisms between these chains. The material of this section is very close to [1, 2, 3].

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In Section 4 we consider a class of finitary codes between topologically transitive sofic systems (Y, S_Σ) with periodic points dense. For such systems the section $\tau_+^0(Y, S_\Sigma)$ is a finitary code that, in a sense, is the unique inverse of $\rho_+^0(Y, S_\Sigma)$. Moreover, $\tau_+^0(Y, S_\Sigma)$ has bounded anticipation and its coding time has moments of all orders. Thus $\rho_+^0(Y, S_\Sigma)$ belongs to a class of finitary codes that was considered in [5] for topological Markov chains. For the case that the dimension groups of the chains $(X_+^0(Y, S_\Sigma), S_+^0(Y, S_\Sigma))$ are totally ordered, we point out, that by the results of [5] the shift equivalence of the chains $(X_+^0(Y, S_\Sigma), S_+^0(Y, S_\Sigma))$ is a necessary and sufficient condition for the existence of such codes.

2. Canonicity of an extension

We consider a sofic system (Y, S_Σ).

(2.1) LEMMA. *Let u be an automorphism of $(X_+(Y, S_\Sigma), S_+(Y, S_\Sigma))$ such that*

$$\rho_+(Y, S_\Sigma)u = \rho_+(Y, S_\Sigma).$$

Then u is the identity.

PROOF. Let

$$(y_i, S_\Sigma^{i-1}E_i)_{i\in\mathbf{Z}} \in X_+(Y, S_\Sigma),$$

and define

$$\bar{E}_i \in \omega_+(Y, S_\Sigma)P_{(-\infty,i)}(Y), \qquad i\in\mathbf{Z}$$

by

$$(y_i, S_\Sigma^{i-1}\bar{E}_i)_{i\in\mathbf{Z}} = u((y_i, S_\Sigma^{i-1}E_i)_{i\in\mathbf{Z}}).$$

We have to show that

$$\bar{E}_i = E_i, \qquad i\in\mathbf{Z}. \tag{1}$$

Let $N\in\mathbf{N}$ be such that u and u^{-1} are given by block maps

$$\Phi : P_{[-N,N]}X_+(Y, S_\Sigma) \to \Omega_+(Y, S_\Sigma),$$

$$\bar{\Phi} : P_{[-N,N]}X_+(Y, S_\Sigma) \to \Omega_+(Y, S_\Sigma).$$

Thus

$$u((y_i, S_\Sigma^{i-1}E_i)_{i\in\mathbf{Z}}) = (\Phi((y_j, S_\Sigma^{j-1}E_j)_{i-N\le j\le i+N}))_{i\in\mathbf{Z}},$$

and, for $i\in\mathbf{Z}$, $\bar{E}_i$ is determined by $y_{[i-N,i+N]}$ and E_{i-N}. Let

$$z^{(+)} \in E_{i+N+1},$$

set

$$y'_j = \begin{cases} y_j, & j \leqq i+N, \\ z_j^{(+)}, & j > i+N, \end{cases}$$

$$E'_j = \begin{cases} E_j, & j \leqq i+N \\ P_{[j,\infty)}(Z(z^{(+)}_{(i+N,j]}) \cap E_j), & j > i+N \end{cases}$$

and define

$$\bar{E}'_j \in \omega_+(Y, S_\Sigma) P_{(-\infty,j)}(Y), \qquad j \in \mathbf{Z}$$

by

$$(y'_j, S_\Sigma^{j-1} \bar{E}'_j)_{j \in \mathbf{Z}} = u((y'_j, S_\Sigma^{j-1} E'_j)_{j \in \mathbf{Z}}).$$

Then

$$\bar{E}'_j = E_j, \qquad j \leqq i,$$

and this implies that

$$\bar{E}'_{i+N+1} = P_{(i+N,\infty)}(Z(y'_{[i,i+N]}) \cap \bar{E}'_i) = \bar{E}_{i+N+1}.$$

It is also $z^{(+)} \in \bar{E}'_{i+N+1}$ and we have therefore

$$\bar{E}_{i+N+1} \supset E_{i+N+1}, \qquad i \in \mathbf{Z}.$$

By symmetry (1) is shown. Q.e.d.

(2.2) THEOREM. *Let*

$$u : (Y, S_\Sigma) \rightarrow (\bar{Y}, S_{\bar{\Sigma}})$$

be a topological conjugacy of sofic systems. Then u_+ *is the unique topological conjugacy of* $(X_+(Y, S_\Sigma), S_+(Y, S_\Sigma))$ *onto* $(X_+(\bar{Y}, S_{\bar{\Sigma}}), S_+(\bar{Y}, S_{\bar{\Sigma}}))$ *such that*

$$u\rho_+(Y, S_\Sigma) = \rho_+(\bar{Y}, S_{\bar{\Sigma}}) u_+.$$

PROOF. This follows from Theorem (2.14) of [4] and Lemma (2.1). Q.e.d.

3. Homomorphisms

Consider a sofic system (Y, S_Σ) that is given as the image of an irreducible topological Markov chain $(X_{\hat{A}}, S_\Sigma)$ under a homomorphism v with finite degree d. Let, for some $I \in \mathbf{Z}_+$, v be given by a block map

$$\Phi : P_{[-I,I]}(X_{\hat{A}}) \to \Sigma.$$

Thus

$$v\hat{x} = (\Phi(\hat{x}_{[i-I,i+I]}))_{i\in\mathbf{Z}}, \qquad \hat{x} \in X_{\hat{A}}.$$

By general arguments one has for all blocks

$$a \in P_{[j,k]}(Y), \qquad j,k \in \mathbf{Z}, \quad j \leqq k$$

that

$$\inf_{j\leqq i\leqq k} |P_{[i-I,i+I]}(v^{-1}Z(a))| \geqq d$$

and there always exists for some j, $k \in \mathbf{Z}$, $j \leqq k$ an $a \in P_{[j,k]}(Y)$ such that

$$\inf_{j\leqq i\leqq k} |P_{[i-I,i+I]}(v^{-1}Z(a))| = d.$$

We call such blocks Φ-finitary. For the next lemma compare Lemma (2.4) of [4] and its proof.

(3.1) Lemma. *Every Φ-finitary block is a finitary block of (Y, S_Σ).*

Proof. Recode $(X_{\hat{A}}, S_\Sigma)$ to make Φ into a 1-block map. Let

$$a \in P_{[j,k]}(Y), \qquad j,k \in \mathbf{Z}, \quad j \leqq k,$$

be a Φ-finitary block, and let i_0, $j \leqq i_0 \leqq k$, be such that

$$|P_{i_0} v^{-1} Z(a)| = d.$$

For all

$$y_- \in P_{(-\infty,k]}(Y) \cap Z(a),$$

one has

$$\omega_+(Y, S_\Sigma)(y_-)$$
$$= \bigcup_{\hat{\sigma}_k \in P_k v^{-1} Z(y_-)} \{\Phi(\hat{\sigma}_i)_{k<i<\infty} : (\hat{\sigma}_i)_{k<i<\infty} \in P_{(k,\infty)}(X_{\hat{A}}), \hat{A}(\hat{\sigma}_k, \hat{\sigma}_{k+1}) = 1\}.$$

Hence the lemma is proved once it is shown that for all

$$y_- \in P_{(-\infty,k]}(Y) \cap Z(a)$$

one has

$$P_k v^{-1}(y_-) = P_k v^{-1} Z(a).$$

Assume that for some

$$y_- \in P_{(-\infty,k]}(Y) \cap Z(a)$$

one has that $P_k\nu^{-1}Z(y_-)$ is a proper subset of $P_k\nu^{-1}(Z(a))$. Then there exists by compactness a $j'<j$ such that one has for

$$a'=(y_-)_{[j',k]}$$

that $P_k\nu^{-1}Z(a')$ is a proper subset of $P_k\nu^{-1}Z(a)$, and it follows that $P_{i_0}\nu^{-1}Z(a')$ is a proper subset of $P_{i_0}\nu^{-1}Z(a)$, for otherwise

$$\begin{aligned}P_{[i_0,k]}\nu^{-1}Z(a')&=\{\hat{a}\in P_{[i_0,k]}X_{\hat{A}}:\Phi(\hat{a}_i)=a_i,\ i_0\leqq i\leqq k,\ a_{i_0}\in P_{i_0}\nu^{-1}Z(a')\}\\&=\{\hat{a}\in P_{[i_0,k]}X_{\hat{A}}:\Phi(\hat{a}_i)=a_i,\ i_0\leqq i\leqq k,\ a_{i_0}\in p_{i_0}\nu^{-1}Z(a)\}\\&=P_{[i_0,k]}\nu^{-1}Z(a).\end{aligned}$$

Thus

$$|P_{i_0}\nu^{-1}Z(a')|<d,$$

which is a contradiction, since d is the degree of ν. Q.e.d.

The next proposition is properly formulated in terms of state transition graphs (labeled directed graphs) as are customarily used to represent sofic systems. Let $(A(\sigma,\sigma'))_{\sigma,\sigma'\in\Sigma}$ be a $\mathbf{Z}_+$-matrix and consider the directed graph with vertex set Σ and $A(\sigma,\sigma')$ arcs with initial vertex σ and final vertex σ'. With this directed graph there is associated the topological Markov chain $(X_A, S_{\Sigma\times\mathbf{N}})$, where

$$X_A=\{(x_i,k_i)_{i\in\mathbf{Z}}\in(\Sigma\times\mathbf{N})^{\mathbf{Z}}:1\leqq k_i\leqq A(x_i,x_{i+1}),\ i\in\mathbf{Z}\}.$$

A state transition graph is obtained by labeling each arc of the graph with a symbol from some symbol set $\bar{\Sigma}$. In other words, one specifies a map

$$\Psi:\{(\sigma,k,\sigma')\in\Sigma\times\mathbf{N}\times\Sigma:1\leqq k\leqq A(\sigma,\sigma')\}\to\bar{\Sigma}.$$

Ψ as a 2-block map gives a homomorphism ν_Ψ of $(X_A, S_{\Sigma\times\mathbf{N}})$ onto the sofic system that is defined by the state transition graph,

$$\nu_\Psi((x_i,k_i)_{i\in\mathbf{Z}})=(\Psi(x_{i-1},k_{i-1},x_i))_{i\in\mathbf{Z}},\qquad (x_i,k_i)_{i\in\mathbf{Z}}\in X_A.$$

The labeling Ψ is said to be 1-right resolving, and the state transition graph (Σ, A, Ψ) is called a Shannon graph if for every $\sigma\in\Sigma$ and for every $\bar{\sigma}\in\bar{\Sigma}$ there is at most one arc with initial vertex σ that is labeled $\bar{\sigma}$. We consider now a Shannon graph (Σ, A, Ψ). Note that ν_Ψ is here a right resolving homomorphism. We denote

$$\omega_+(\Psi)(\sigma)=\{(\Psi(\sigma_i,k_i,\sigma_{i+1}))_{i\in\mathbf{Z}_+}:(\sigma_i,k_i)_{i\in\mathbf{Z}_+}\in P_{[0,\infty)}(X_A),\ \sigma_0=\sigma\}.$$

We say that $\sigma,\sigma'\in\Sigma$ are Ψ-equivalent (more precisely $\Psi(+)$-equivalent) if

$\omega_+(\Psi)(\sigma)$ is equal to $\omega_+(\Psi)(\sigma')$. The set Σ_Ψ of Ψ-equivalence classes is the vertex set of a Shannon graph in which there is an arc with initial vertex the Ψ-equivalence class of σ that is labeled $\bar{\sigma}$ if and only if there is an arc in (Σ, A, Ψ) with initial vertex σ that is labeled $\bar{\sigma}$. We denote the matrix of this state transition graph by A_Ψ and its labeling by $\tilde{\Psi}$. Observe that all $\tilde{\Psi}$-equivalence classes consist of one element of Σ_Ψ.

Set

$$P_{\Sigma,i}((x_i, k_i)_{i\in\mathbf{Z}}) = x_i, \qquad (x_i, k_i)_{i\in\mathbf{Z}} \in (\Sigma \times \mathbf{N})^{\mathbf{Z}}.$$

(3.2) Proposition. *Let (Σ, A, Ψ) be an irreducible Shannon graph, and let*

$$\nu_\Psi : (X_A, S_{\Sigma\times\mathbf{N}}) \to (\bar{Y}, S_\Sigma).$$

Then a block

$$a \in P_{[j,k]}(\bar{Y}), \qquad j \leqq k, \quad j, k \in \mathbf{Z}$$

is finitary for $(\bar{Y}, S_\Sigma)$ if and only if all elements of $P_{\Sigma,k}\nu_\Psi^{-1}Z(a)$ are Ψ-equivalent.

Proof. If all elements of $P_{\Sigma,k}\nu_\Psi^{-1}Z(a)$ are Ψ-equivalent, then a is a finitary block for $(\bar{Y}, S_\Sigma)$. Indeed, then

$$\omega_+(Y, S_\Sigma)(a) = \omega_+(\Psi)(\sigma), \qquad \sigma \in P_{\Sigma,k}\nu_\Psi^{-1}Z(a).$$

On the other hand, if a is a finitary block for $(\bar{Y}, S_\Sigma)$ and also is Ψ-finitary then the right resolving property of Ψ implies that all elements of $P_{\Sigma,k}\nu_\Psi^{-1}Z(a)$ are Ψ-equivalent. Assume then that a is not Ψ-finitary, and that there are

$$\sigma, \sigma' \in P_{\Sigma,k}\nu_\Psi^{-1}Z(a)$$

that are not Ψ-equivalent. The Shannon graph being irreducible there exist then for some $J < j$ Ψ-finitary blocks

$$b, b' \in P_{[J,k]}(Y) \cap Z(a)$$

that are then, by Lemma (3.1), also finitary for (Y, S_Σ), such that all elements of $P_{\Sigma,k}\nu_\Psi^{-1}Z(b)$ are Ψ-equivalent to σ and all elements of $P_{\Sigma,k}\nu_\Psi^{-1}Z(b')$ are Ψ-equivalent to σ'. Then

$$\omega_+(\bar{Y}, S_\Sigma)(b) = \omega_+(\Psi)(\sigma) \neq \omega_+(\Psi)(\sigma') = \omega_+(\bar{Y}, S_\Sigma)(b'),$$

and a is not finitary for $(\bar{Y}, S_\Sigma)$. Q.e.d.

As we have seen, given an irreducible Shannon graph (Σ, A, Ψ),

$$\nu_\Psi : (X_A, S_{\Sigma\times\mathbf{N}}) \to (\bar{Y}, S_\Sigma),$$

and a finitary block $a \in P_{[j,0]}(\bar{Y})$, $j \leqq 0$, for $(\bar{Y}, S_{\bar{\Sigma}})$, one has

$$\omega_+(\bar{Y}, S_{\bar{\Sigma}})(a) = \omega_+(\Psi)(\sigma), \quad \sigma \in P_{\Sigma,0} \nu_\Psi^{-1} Z(a).$$

We can therefore identify $\Xi_+^0(\bar{Y}, S_{\bar{\Sigma}})$ with Σ_Ψ and ν_Ψ is then a presentation of $\rho_+^0(\bar{Y}, S_{\bar{\Sigma}})$.

(3.3) THEOREM. *The following are equivalent for a sofic system* (Y, S_Σ):

(a) (Y, S_Σ) *is the homomorphic image of an irreducible topological Markov chain under a bi-resolving homomorphism.*

(b) (Y, S_Σ) *is topological transitive with periodic points dense and the homomorphism* $\rho_+^0(Y, S_\Sigma)$ *is left-resolving.*

PROOF. We assume (a) and prove (b). After recoding one has an irreducible state transition graph $(\hat{\Sigma}, A, \Psi)$ whose labeling Ψ is 1-bi-resolving, and

$$\nu_\Psi : (X_A, S_{\hat{\Sigma} \times \mathbf{N}}) \to (Y, S_\Sigma).$$

Since $(\hat{\Sigma}, A)$ is irreducible and Ψ is 1-bi-resolving it follows that $\Psi(+)$-equivalence and $\Psi(-)$-equivalence are the same. Identifying $\Xi_0^+(Y, S_\Sigma)$ as well as $\Xi_0^-(Y, S_\Sigma)$ with Σ_Ψ, and setting

$$w = S_{\Sigma_\Psi} \times 1$$

one has a commutative diagram

$$(X_-^0(Y, S_\Sigma), S_-^0(Y, S_\Sigma)) \overset{w}{\leftrightarrow} (X_+^0(Y, S_\Sigma), S_+^0(Y, S_\Sigma))$$
$$\rho_-^0(Y, S_\Sigma) \searrow \qquad \swarrow \rho_+^0(Y, S_\Sigma)$$
$$(Y, S_\Sigma)$$

Q.e.d.

Given a homomorphism

$$\nu : (Y, S_\Sigma) \to (\bar{Y}, S_{\bar{\Sigma}})$$

of sofic systems one can form the fiber product $(X_+(\nu), S_+(\nu))$ of $(X_+(Y, S_\Sigma), S_+(Y, S_\Sigma))$ and $(X_+(\bar{Y}, S_{\bar{\Sigma}}), S_+(\bar{Y}, S_{\bar{\Sigma}}))$ with respect to $\nu, \rho_+(Y, S_\Sigma)$ and $\rho_+(\bar{Y}, S_{\bar{\Sigma}})$, and has then the projections

$$\pi_+(\nu) : (X_+(\nu), S_+(\nu)) \to (X_+(Y, S_\Sigma), S_+(Y, S_\Sigma)),$$

$$\bar{\pi}_+(\nu) : (X_+(\nu), S_+(\nu)) \to (X_+(\bar{Y}, S_{\bar{\Sigma}}), S_+(\bar{Y}, S_{\bar{\Sigma}})).$$

Similarly one has the fiber product $(X_-(\nu), S_-(\nu))$ of $(X_-(X, S_\Sigma), S_+(Y, S_\Sigma))$

and $(X_-(\bar{Y}, S_\Sigma), S_-(\bar{Y}, S_\Sigma))$ with respect to $\nu\rho_-(\bar{Y}, S_\Sigma)$ and $\rho_-(\bar{Y}, S_\Sigma)$ and projections $\pi_-(\nu)$ and $\bar{\pi}_-(\nu)$. The commutative diagram

$$\begin{array}{ccccc}
 & (X_-(Y,S_\Sigma),S_-(Y,S_\Sigma)) \xrightarrow{\rho_-(Y,S_\Sigma)} & (Y,S_\Sigma) & \xleftarrow{\rho_+(Y,S_\Sigma)} (X_+(Y,S_\Sigma),S_+(Y,S_\Sigma)) & \\
{}^{\pi_-(\nu)}\nearrow & & \Big| & & \nwarrow{}^{\pi_+(\nu)} \\
(X_-(\nu),S_-(\nu)) & & \Big|\,\nu & & (X_+(\nu),S_+(\nu)) \\
\searrow{}^{\bar{\pi}_-(\nu)} & & \Big\downarrow & & {}^{\bar{\pi}_+(\nu)}\swarrow \\
 & (X_-(\bar{Y},S_\Sigma),S_-(\bar{Y},S_\Sigma)) \xrightarrow{\rho_-(\bar{Y},S_\Sigma)} & (\bar{Y},S_\Sigma) & \xleftarrow{\rho_+(\bar{Y},S_\Sigma)} (X_+(\bar{Y},S_\Sigma),S_+(\bar{Y},S_\Sigma)) &
\end{array}$$

can be completed by forming in addition the fiber product of $(X_-(\nu), S_-(\nu))$ and $(X_+(\nu), S_+(\nu))$ with respect to $\pi_-(\nu)$ and $\pi_+(\nu)$, that projects then also onto $(X_{-+}(Y, S_\Sigma), S_{-+}(Y, S_\Sigma))$ and $(X_{-+}(\bar{Y}, S_\Sigma), S_{-+}(\bar{Y}, S_\Sigma))$. If here ν is given by a 1-block map Φ then the state space of $(X_+(\nu), S_+(\nu))$ is given by

$$\{(\sigma, E, \bar{E}) \in \Sigma \times \Xi_+(Y, S_\Sigma) \times \Xi_+(\bar{Y}, S_\Sigma) : Z(\sigma) \cap E \neq \emptyset,\ Z(\Phi(\sigma)) \cap \bar{E} \neq \emptyset\}$$

and a transition from an element $(\sigma, E, \bar{E})$ of this state space to another element $(\sigma', E', \bar{E}')$ is allowed precisely if

$$E' = S_\Sigma P_{(1,\infty)}(Z(\sigma) \cap E) \quad \text{and} \quad \bar{E}' = S_\Sigma P_{(1,\infty)}(Z(\Phi(\sigma)) \cap \bar{E}).$$

Some of the properties of ν are reflected in the diagram. E.g., ν is right resolving if and only if $\bar{\pi}_+(\nu)$ is right resolving. Also, if (Y, S_Σ) is topologically transitive with periodic points dense, and if ν is right resolving, then (Y, S_Σ) is Markov precisely if $\rho_+(Y, S_\Sigma)\pi_+(\nu)$, when restricted to the appropriate irreducible component of $(X_+(\nu), S_+(\nu))$, becomes a topological conjugacy. Indeed, let (Y, S_Σ) be Markov, and recode such that one has an irreducible Shannon graph (Σ, A, Ψ) and $\nu = \nu_\Psi$. Denote by $[\sigma]_\Psi$ the Ψ-equivalence class of a $\sigma \in \Sigma$, and define a shift-commuting map $\beta(\Psi)$ of $(X_A, S_{\Sigma \times \mathbf{N}})$ into $(X_+(\nu), S_+(\nu))$ by

$$\beta(\Psi)((\sigma_i, k_i)_{i \in \mathbf{Z}}) = (\Psi(\sigma_{i-1}, k_{i-1}, \sigma_i), (\sigma_i, k_i), [\sigma_{i-1}]_\Psi)_{i \in \mathbf{Z}}, \qquad (\sigma_i, k_i)_{i \in \mathbf{Z}} \in X_A.$$

$\beta(\Psi)$ is a topological conjugacy of $(X_A, S_{\Sigma \times \mathbf{N}})$ onto an irreducible component of $(X_+(\nu), S_+(\nu))$, whose inverse is the appropriately restricted $\pi_+(\nu)$. Note also that $\bar{\pi}_+(\nu)\beta(\Psi)$ is a presentation of a canonical homomorphism of the topological Markov chain (Y, S_Σ) onto the irreducible canonical extension of $(Y, \hat{S}_\Sigma)$.

4. A class of finitary codes

We consider topologically transitive sofic systems (Y, S_Σ), $(\bar{Y}, S_{\bar{\Sigma}})$ with periodic points dense. Let μ_Y be the measure of maximal entropy on (Y, S_Σ). We consider finitary codes v between (Y, S_Σ) and $(\bar{Y}, S_{\bar{\Sigma}})$. By this is meant a shift commuting densely defined and μ_Y-a.e. continuous map of Y into $\bar{Y}$. To be definite we let the domain of definition D_u of u be the maximal $\mathscr{G}_\delta$ to which it can be extended by continuity. We require that v has bounded anticipation and that its coding time has moments of all orders. This means that there is an $I \in \mathbf{N}$ and a map $x \to i(x) \leqq 0$ $(x \in D_u)$ such that $(ux)_0$ is determined by $x_{[-i(x), I]}$, and such that

$$\int_Y (i(x))^p d\mu_Y(x) < \infty, \qquad p \geqq 1.$$

We remark at this point that for all codes that are constructed in this context one has actually that for some $\alpha > 0$

$$\mu_Y\{x \in D_u : i(x) \geqq n\} \leqq e^{-\alpha n}, \qquad n \in \mathbf{N}.$$

We call the code u resolving, if all points y in D_u, except those in a μ_Y-nullset of the remote past, are uniquely determined by uy together with any of the initial segments $y_{(-\infty, i]}$, $i \in \mathbf{Z}$.

(4.1) THEOREM. *Let (Y, S_Σ), $(\bar{Y}, S_{\bar{\Sigma}})$ be topologically transitive sofic systems with periodic points dense and equal entropy* $\log \lambda$. *With $f_\lambda(z)$ the minimal polynomial of λ, and let the inverse of the zeta function of $(X^0_+(\bar{Y}, S_{\bar{\Sigma}}), S^0_+(Y, S_\Sigma))$ and $(X^0_+(\bar{Y}, S_{\bar{\Sigma}}), S^0_+(\bar{Y}, S_{\bar{\Sigma}}))$ be $f_\lambda(z^{-1})$ (up to a power of z). Then the following are equivalent:*

(a) *There exists a μ_Y-a.e. finite-to-one resolving finitary code between (Y, S_Σ) and $(\bar{Y}, S_{\bar{\Sigma}})$ with bounded anticipation whose coding time has moments of all orders.*

(b) *There exists a μ_Y-a.e. one-to-one finitary code between (Y, S_Σ) and $(\bar{Y}, S_{\bar{\Sigma}})$ with bounded anticipation whose coding time has moments of all orders.*

(c) *$(X^0_+(Y, S_\Sigma), S^0_+(Y, S_\Sigma))$ and $(X^0_+(\bar{Y}, S_{\bar{\Sigma}}), S^0_+(\bar{Y}, S_{\bar{\Sigma}}))$ are shift equivalent.*

PROOF. It suffices to observe that the sections $\tau^0_+(Y, S_\Sigma)$ are finitary codes with bounded anticipation whose coding times have moments of all orders, and to apply Theorem (4.2) of [5].

(4.2) COROLLARY. *Let (Y, S_Σ) and $(\bar{Y}, S_{\bar{\Sigma}})$ be topologically transitive sofic systems with periodic points dense and equal entropy* $\log \lambda$. *With $f_\lambda(z)$ the* *minimal polynomial of* λ, *let the inverse of the zeta function of* $(X_+^0(Y, S_\Sigma), S_+^0(Y, S_\Sigma))$ *and* $(X_+^0(\bar{Y}, S_{\bar{\Sigma}}), S_+^0(\bar{Y}, S_{\bar{\Sigma}}))$ *be (up to a power of* z*)* $f_\lambda(z^{-1})$. *If there exists a resolving homomorphism of* (Y, S_Σ) *onto* $(\bar{Y}, S_{\bar{\Sigma}})$ *then* $(X_+^0(Y, S_\Sigma), S_+^0(Y, S_\Sigma))$ *and* $(X_+^0(\bar{Y}, S_{\bar{\Sigma}}), S_+^0(\bar{Y}, S_{\bar{\Sigma}}))$ *are shift equivalent.*

PROOF. A resolving homomorphism is a finite-to-one resolving finitary code that has bounded anticipation and bounded memory. Q.e.d.

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